A Generalized (G'/G)-Expansion Method for the Nonlinear Schrödinger Equation with Variable Coefficients

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In this paper, a generalized (G'/G)-expansion method, combined with suitable transformations, is used to construct exact solutions of the nonlinear Schrödinger equation with variable coefficients. As a result, hyperbolic function solutions, trigonometric function solutions, and rational solutions with parameters are obtained. When the parameters are taken as special values, some solutions including the known kink-type solitary wave solution and the singular travelling wave solution are derived from these obtained solutions. It is shown that the generalized (G'/G)-expansion method is direct, effective, and can be used for many other nonlinear evolution equations with variable coefficients in mathematical physics.

Key words: Nonlinear Evolution Equations; Generalized (G'/G)-Expansion Method; Hyperbolic Function Solutions; Trigonometric Function Solutions; Rational Solutions.

1. Introduction

Seeking exact solutions of nonlinear evolution equations (NLEEs) is of important significance in mathematical physics and becomes one of the most exciting and extremely active areas of research investigation. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine-cosine method [5], homogeneous balance method [6], homotopy perturbation method [7], variational method [8], asymptotic methods [9], Adomian decomposition method [10], mapping approach [11], tanh-function method [12 – 15], algebraic method [16, 17], Jacobi elliptic function expansion method [18, 19], F-expansion method [20 – 22], auxiliary equation method [23 – 25], and Exp-function method [26 – 28].

However, to our knowledge, most of aforementioned methods are related to the constant-coefficient models. Recently, the study of variable-coefficient NLEEs has attracted much attention [8, 28 – 32] because most of real nonlinear physical equations possess variable coefficients. Wang et al. [33] proposed a new method called the $(\frac{G'}{G})$-expansion method to look for travelling wave solutions of NLEEs. The $(\frac{G'}{G})$-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$, and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$G'' + \lambda G' + \mu G = 0, \tag{1}$$

where $G' = \mathrm{d}G(\xi)/\mathrm{d}\xi$, $G'' = \mathrm{d}^2G(\xi)/\mathrm{d}\xi^2$, $\xi = x - Vt$, and V is a constant. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the $(\frac{G'}{G})$-expansion method, Wang et al. [33, 34] and Bekir [35] have successfully obtained more travelling wave solutions of some important NLEEs. More recently, Zhang et al. [36, 37] proposed a generalized $(\frac{G'}{G})$-expansion method to improve and extend Wang et al.'s work [33] for solving variable-coefficient equations and high-dimensional equations.

The present paper is motivated by the desire to further extend the generalized $(\frac{G'}{G})$-expansion method [36] to other types of NLEEs. With this purpose, we will consider the nonlinear Schrödinger equation with variable coefficients:

$$\mathrm{i}u_z + \frac{1}{2}\beta(z)u_{tt} + \alpha(z)|u|^2u = \mathrm{i}\gamma(z)u, \tag{2}$$

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where $u = u(z,t)$ is a real or complex valued function of z and t, $\alpha(z)$, $\beta(z)$, and $\gamma(z)$ are all arbitrary functions of the indicated variable.

The rest of this paper is organized as follows: In Section 2, we briefly describe the generalized $(\frac{G'}{G})$-expansion method. In Section 3, we use the generalized $(\frac{G'}{G})$-expansion method and several suitable transformations to solve equation (2). In Section 4, some conclusions are given.

2. Description of the Generalized $(\frac{G'}{G})$-Expansion Method

For a given NLEE with independent variables $X = (x, y, z, \ldots, t)$ and dependent variable u:

$$F(u, u_t, u_x, u_y, u_z, \ldots, u_{xt}, u_{yt}, u_{zt} \ldots, u_{tt}, u_{xx}, u_{yy}, u_{zz}, \ldots) = 0, \tag{3}$$

we suppose its solution can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u = \sum_{i=1}^{m} \alpha_i(X) \left(\frac{G'}{G}\right)^i + \alpha_0(X), \quad \alpha_m(X) \neq 0, \tag{4}$$

where $\alpha_0(X)$, $\alpha_i(X)$ $(i = 1, 2, \cdots, m)$, and $\xi = \xi(X)$ are all functions of X to be determined later, $G = G(\xi)$ satisfies equation (1). To determine u explicitly, we take the following four steps:

Step 1. Determine the integer m by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in (3).

Step 2. Substitute (4) along with (1) into (3) and collect all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (3) is converted into a polynomial in $(\frac{G'}{G})$. Then set each coefficient of this polynomial to zero to derive a set of over-determined partial differential equations for $\alpha_0(X)$, $\alpha_i(X)$, and ξ.

Step 3. Solve the system of over-determined partial differential equations obtained in Step 2 for $\alpha_0(X)$, $\alpha_i(X)$, and ξ by use of *Mathematica*.

Step 4. Use the results obtained in the above steps to derive a series of fundamental solutions of (3) depending on $(\frac{G'}{G})$, since the solutions of equation (1) have been well known for us, then we can obtain exact solutions of (3).

3. Exact Solutions of Equation (2)

In order to obtain the exact solution of equation (2), firstly we make the transformation

$$u(z,t) = V(z,t)\exp\left(\mathrm{i}\theta(z,t)\right), \tag{5}$$

where $V(z,t)$ and $\theta(z,t)$ are amplitude and phase functions, respectively. Substituting (5) into (2) and separating the real and imaginary parts, we obtain

$$-V\theta_z + \frac{1}{2}\beta(z)(V_{tt} - V\theta_t^2) + \alpha(z)V^3 = 0, \tag{6}$$

$$V_z + \frac{1}{2}\beta(z)(2V_t\theta_t + V\theta_{tt}) - \gamma(z)V = 0. \tag{7}$$

Balancing V_{tt} and V^3 in (6), we have $m = 1$. We assume that (6) and (7) have the following formal solutions:

$$V = \alpha_1(z)\left(\frac{G'}{G}\right) + \alpha_0(z), \quad \alpha_1(z) \neq 0, \tag{8}$$

$$\theta = \Phi(z)t^2 + \Gamma(z)t + \Omega(z), \tag{9}$$

where $G = G(\xi)$ satisfies equation (1), $\xi = p(z)t + q(z)$, $\alpha_0(z)$, $\alpha_1(z)$, $\Phi(z)$, $\Gamma(z)$, $\Omega(z)$, $p(z)$, and $q(z)$ are differentiable functions to be determined.

Substituting (8) and (9) into (6) and (7), and collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand sides of (6) and (7) are converted into two polynomials in $t^j(\frac{G'}{G})$ $(j = 0, 1, 2)$. Setting each coefficient of the polynomials to zero, we derive a set of over-determined differential equations for $\alpha_0(z)$, $\alpha_1(z)$, $\Phi(z)$, $\Gamma(z)$, $\Omega(z)$, $p(z)$, and $q(z)$ as follows:

$$t^0(\frac{G'}{G})^0: \quad \alpha(z)\alpha_0^3(z) + \frac{1}{2}\lambda\mu p^2(z)\alpha_1(z)\beta(z) - \frac{1}{2}\alpha_0(z)\beta(z)\Gamma^2(z) - \alpha_0(z)\Omega'(z) = 0,$$

$$t^0(\frac{G'}{G})^1: \quad 3\alpha(z)\alpha_0^2(z)\alpha_1(z) + \frac{1}{2}\lambda^2 p^2(z)\alpha_1(z)\beta(z) + \mu p^2(z)\alpha_1(z)\beta(z) - \frac{1}{2}\alpha_1(z)\beta(z)\Gamma^2(z) - \alpha_1(z)\Omega'(z) = 0,$$

$$t^0(\frac{G'}{G})^2: \quad 3\alpha(z)\alpha_0(z)\alpha_1^2(z) + \frac{3}{2}\lambda p^2(z)\alpha_1(z)\beta(z) = 0,$$

$$t^0(\frac{G'}{G})^3: \quad \alpha(z)\alpha_1^3(z) + p^2(z)\alpha_1(z)\beta(z) = 0,$$

$$t^1(\frac{G'}{G})^0: \quad -2\alpha_0(z)\beta(z)\Gamma(z)\Phi(z) - \alpha_0(z)\Gamma'(z) = 0,$$

$$t^1(\frac{G'}{G})^1: \quad -2\alpha_1(z)\beta(z)\Gamma(z)\Phi(z) - \alpha_1(z)\Gamma'(z) = 0,$$

$$t^2(\frac{G'}{G})^0: \quad -2\alpha_0(z)\beta(z)\Phi^2(z) - \alpha_0(z)\Phi'(z) = 0,$$

$$t^2(\frac{G'}{G})^1: \quad -2\alpha_1(z)\beta(z)\Phi^2(z) - \alpha_1(z)\Phi'(z) = 0,$$

$$t^0(\frac{G'}{G})^0: \quad -\alpha_0(z)\gamma(z) - \mu p(z)\alpha_1(z)\beta(z)\Gamma(z) + \alpha_0(z)\beta(z)\Phi(z) - \mu\alpha_1(z)q'(z) + \alpha_0'(z) = 0,$$

$$t^0(\frac{G'}{G})^1: \quad -\alpha_1(z)\gamma(z) - \mu p(z)\alpha_1(z)\beta(z)\Gamma(z) + \alpha_1(z)\beta(z)\Phi(z) - \mu\alpha_1(z)q'(z) + \alpha_1'(z) = 0,$$

$$t^0(\frac{G'}{G})^2: \quad -p(z)\alpha_1(z)\beta(z)\Gamma(z) - \alpha_1(z)q'(z) = 0,$$

$$t^1(\frac{G'}{G})^0: \quad -2\mu p(z)\alpha_1(z)\beta(z)\Phi(z) - \mu\alpha_1(z)p'(z) = 0,$$

$$t^1(\frac{G'}{G})^1: \quad -2\lambda p(z)\alpha_1(z)\beta(z)\Phi(z) - \lambda\alpha_1(z)p'(z) = 0,$$

$$t^1(\frac{G'}{G})^2: \quad -2p(z)\alpha_1(z)\beta(z)\Phi(z) - \alpha_1(z)p'(z) = 0.$$

Solving the set of over-determined partial differential equations by use of *Mathematica*, we get the following two cases:

Case 1.

$$\alpha_0(z) = \frac{1}{2}A_1\lambda\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right),$$
$$\alpha_1(z) = A_1\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right),$$

$$\Phi(z) = 0, \quad p(z) = A_2, \quad \Gamma(z) = A_3,$$
$$q(z) = -A_2A_3\int^z \beta(\zeta)\mathrm{d}\zeta + A_4,$$

$$\Omega(z) = -\frac{1}{4}[2A_3^2 + A_2^2(\lambda^2 - 4\mu)]\int^z \beta(\zeta)\mathrm{d}\zeta + A_5,$$
$$\beta(z) = -\frac{A_1^2}{A_2^2}\alpha(z)\exp\left(2\int^z \gamma(\zeta)\mathrm{d}\zeta\right),$$

where A_1, A_2, A_3, A_4, and A_5 are arbitrary constants.

Case 2.

$$\alpha_0(z) = \frac{1}{2}A_6\lambda\Phi^{\frac{1}{2}}(z)\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right),$$
$$\alpha_1(z) = A_6\Phi^{\frac{1}{2}}(z)\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right),$$

$$\Phi(z) = \Phi(z), \quad p(z) = A_7\Phi(z), \quad \Gamma(z) = A_8\Phi(z),$$
$$q(z) = \frac{1}{2}A_7A_8\Phi(z) + A_9,$$
$$\Omega(z) = \frac{1}{8}[2A_8^2 + A_7^2(\lambda^2 - 4\mu)]\Phi(z) + A_{10},$$
$$\alpha(z) = \frac{A_7^2\Phi'(z)\exp\left(-2\int^z \gamma(\zeta)\mathrm{d}\zeta\right)}{2A_6^2\Phi(z)},$$
$$\beta(z) = -\frac{\Phi'(z)}{2\Phi^2(z)},$$

where A_6, A_7, A_8, A_9, and A_{10} are arbitrary constants, $\Phi(z)$ is an arbitrary function of z and $\Phi'(z) = \mathrm{d}\Phi(z)/\mathrm{d}z$.

Substituting Case 1 into (8) and (9), and using (5), we have the first fundamental solution of equation (2):

$$u = \left[A_1\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right)\left(\frac{G'}{G}\right) + \frac{1}{2}A_1\lambda\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right)\right]\exp[\mathrm{i}(A_3t + \Omega(z))], \tag{10}$$

where $G = G(\xi)$, $\xi = A_2t - A_2A_3\int^z \beta(\zeta)\mathrm{d}\zeta + A_4$, $\Omega(z) = -\frac{1}{4}[2A_3^2 + A_2^2(\lambda^2 - 4\mu)]\int^z \beta(\zeta)\mathrm{d}\zeta + A_5$, $\beta(z) = -\frac{A_1^2}{A_2^2}\alpha(z)\cdot\exp\left(2\int^z \gamma(\zeta)\mathrm{d}\zeta\right)$.

Substituting the general solutions of equation (1) into (10), we have three types of exact solutions of equation (2) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain a hyperbolic function solution:

$$u = \frac{1}{2}A_1\sqrt{\lambda^2 - 4\mu}\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right) \cdot\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right) \cdot\exp[\mathrm{i}(A_3t + \Omega(z))], \tag{11}$$

where $\xi = A_2t - A_2A_3\int^z \beta(\zeta)\mathrm{d}\zeta + A_4$, $\Omega(z) = -\frac{1}{4}[2A_3^2 + A_2^2(\lambda^2 - 4\mu)]\int^z \beta(\zeta)\mathrm{d}\zeta + A_5$, $\beta(z) =$

$-\frac{A_1^2}{A_2^2}\alpha(z)\exp\left(2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)$, C_1 and C_2 are arbitrary constants.

If C_1, C_2, λ, and μ are taken as special values, some known results can be rediscovered. For example, setting $C_2=0$, $\mu=\frac{\lambda^2-4}{4}$, $A_1=A_{11}\sqrt{-\frac{c_2}{2c_4}}$, $A_2=\sqrt{-c_2}A_{12}$, $A_4=0$, $A_5=0$, $c_2<0$, and $c_4>0$, then solution (11) can be written as

$$u=A_{11}\sqrt{-\frac{c_2}{2c_4}}\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\tanh(\xi)\cdot\exp[\mathrm{i}(A_3t+\Omega(z))], \tag{12}$$

where $\xi=\sqrt{-c_2}A_{12}(t-A_3\int^z\beta(\zeta)\mathrm{d}\zeta)$, $\Omega(z)=(c_2A_{12}^2-\frac{1}{2}A_3^2)\int^z\beta(\zeta)\mathrm{d}\zeta$, $\beta(z)=-[A_{11}^2/2A_{12}^2c_4]\alpha(z)\exp\left(2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)$. Solution (12) is the known kink-type solitary wave solution (15) found by Zong et al. in [38].

When $\lambda^2-4\mu<0$, we have a trigonometric function solution:

$$u=\frac{1}{2}A_1\sqrt{4\mu-\lambda^2}\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\cdot\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)\cdot\exp[\mathrm{i}(A_3t+\Omega(z))], \tag{13}$$

where $\xi=A_2t-A_2A_3\int^z\beta(\zeta)\mathrm{d}\zeta+A_4$, $\Omega(z)=-\frac{1}{4}[2A_3^2+A_2^2(\lambda^2-4\mu)]\int^z\beta(\zeta)\mathrm{d}\zeta+A_5$, $\beta(z)=-[A_1^2/A_2^2]\alpha(z)\exp\left(2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)$, C_1 and C_2 are arbitrary constants.

If we set $C_2=0$, $\mu=\frac{\lambda^2+4}{4}$, $A_1=-A_{13}$, then solution (13) becomes

$$u=A_{13}\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\tan(\xi)\cdot\exp\left[\mathrm{i}\left(A_3t+\frac{1}{2}(2A_2^2-A_3^2)\int^z\beta(\zeta)\mathrm{d}\zeta+A_5\right)\right], \tag{14}$$

where $\xi=A_2t-A_2A_3\int^z\beta(\zeta)\mathrm{d}\zeta+A_4$, $\beta(z)=-\frac{A_{13}^2}{A_2^2}\alpha(z)\exp\left(2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)$. Solution (14) is equivalent to the trigonometric function solution (37) given by Zhang et al. in [39].

When $\lambda^2-4\mu=0$, we get a rational solution:

$$u=A_1\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\left(\frac{C_2}{C_1+C_2\xi}\right)\cdot\exp[\mathrm{i}(A_3t+\Omega(z))], \tag{15}$$

where $\xi=A_2t-A_2A_3\int^z\beta(\zeta)\mathrm{d}\zeta+A_4$, $\Omega(z)=-\frac{1}{2}A_3^2\int^z\beta(\zeta)\mathrm{d}\zeta+A_5$, $\beta(z)=-[A_1^2/A_2^2]\alpha(z)\exp\left(2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)$, C_1 and C_2 are arbitrary constants.

Substituting Case 2 into (8) and (9), and using (5), we have the second fundamental solution of equation (2):

$$u=\left[A_6\Phi^{\frac{1}{2}}(z)\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\left(\frac{G'}{G}\right)+\frac{1}{2}A_6\lambda\Phi^{\frac{1}{2}}(z)\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\right]\cdot\exp[\mathrm{i}(\Phi(z)t^2+A_8\Phi(z)t+\Omega(z))], \tag{16}$$

where $G=G(\xi)$, $\xi=A_7\Phi(z)t+\frac{1}{2}A_7A_8\Phi(z)+A_9$, $\Omega(z)=\frac{1}{8}[2A_8^2+A_7^2(\lambda^2-4\mu)]\Phi(z)+A_{10}$, $\alpha(z)=A_7^2\Phi'(z)\exp\left(-2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)/2A_6^2\Phi(z)$, $\beta(z)=-\Phi'(z)/2\Phi^2(z)$.

Substituting the general solutions of (1) into (16), we have other three types of exact solutions of equation (2) as follows:

When $\lambda^2-4\mu>0$, we obtain a hyperbolic function solution:

$$u=\frac{1}{2}A_6\sqrt{\lambda^2-4\mu}\Phi^{\frac{1}{2}}(z)\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\cdot\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)\cdot\exp[\mathrm{i}(\Phi(z)t^2+A_8\Phi(z)t+\Omega(z))], \tag{17}$$

where $\xi=A_7\Phi(z)t+\frac{1}{2}A_7A_8\Phi(z)+A_9$, $\Omega(z)=\frac{1}{8}[2A_8^2+A_7^2(\lambda^2-4\mu)]\Phi(z)+A_{10}$, $\alpha(z)=A_7^2\Phi'(z)\exp(-2\int^z\gamma(\zeta)\mathrm{d}\zeta)]/2A_6^2\Phi(z)$, $\beta(z)=-\Phi'(z)/2\Phi^2(z)$, C_1 and C_2 are arbitrary constants.

If we set $C_1=0$, $\mu=\frac{\lambda^2-4}{4}$, then solution (17) gives

$$u=A_6\Phi^{\frac{1}{2}}(z)\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\coth(\xi)\cdot\exp[\mathrm{i}(\Phi(z)t^2+A_8\Phi(z)t+\Omega(z))], \tag{18}$$

where $\xi=A_7\Phi(z)t+\frac{1}{2}A_7A_8\Phi(z)+A_9$, $\Omega(z)=\frac{1}{4}[A_8^2+2A_7^2]\Phi(z)+A_{10}$, $\alpha(z)=A_7^2\Phi'(z)\exp\left(-2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)/2A_6^2\Phi(z)$, $\beta(z)=-\Phi'(z)/2\Phi^2(z)$. Solution (18) is a singular travelling wave solution.

When $\lambda^2 - 4\mu < 0$, we have a trigonometric function solution:

$$u = \frac{1}{2}A_6\sqrt{4\mu-\lambda^2}\Phi^{\frac{1}{2}}(z)\exp\left(\int^z \gamma(\zeta)\mathrm{d}\zeta\right) \cdot \left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right) \cdot \exp\mathrm{i}(\Phi(z)t^2+A_8\Phi(z)t+\Omega(z))], \tag{19}$$

where $\xi = A_7\Phi(z)t+\frac{1}{2}A_7A_8\Phi(z)+A_9$, $\Omega(z)=\frac{1}{8}[2A_8^2 + A_7^2(\lambda^2-4\mu)]\Phi(z)+A_{10}$, $\alpha(z) = A_7^2\Phi'(z)\exp\left(-2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)/2A_6^2\Phi(z)$, $\beta(z)=-\Phi'(z)/2\Phi^2(z)$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$, we get a rational solution:

$$u = A_6\Phi^{\frac{1}{2}}(z)\exp\left(\int^z\gamma(\zeta)\mathrm{d}\zeta\right)\left(\frac{C_2}{C_1+C_2\xi}\right) \cdot \exp[\mathrm{i}(\Phi(z)t^2+A_8\Phi(z)t+\Omega(z))], \tag{20}$$

where $\xi = A_7\Phi(z)t+\frac{1}{2}A_7A_8\Phi(z)+A_9$, $\Omega(z) = \frac{1}{4}A_8^2\Phi(z)+A_{10}$, $\alpha(z)=A_7^2\Phi'(z)\exp\left(-2\int^z\gamma(\zeta)\mathrm{d}\zeta\right)/2A_6^2\Phi(z)$, $\beta(z)=-\Phi'(z)/2\Phi^2(z)$, C_1 and C_2 are arbitrary constants.

Remark 1. All solutions presented in this paper have been checked with *Mathematica* by putting them back into the original equation (2).

Remark 2. To the best of our knowledge, solutions (11), (13), (15), (17) – (20) are new and they have not been reported in [23, 38 – 45].

4. Conclusion

Based on several suitable transformations, the generalized $(\frac{G'}{G})$-expansion method has been successfully used to obtain some new exact solutions with parameters of the nonlinear Schrödinger equation with variable coefficients. The free parameters $A_1 - A_{10}$, C_1, C_2, λ, and μ, especially the arbitrary function $\Phi(z)$ in solutions (17) – (20), can make us discuss the behaviours of solutions and also provide us enough freedom to construct solutions that may be related to real physical problem. It may be of importance in the explanation of some practical problems in physics. The paper shows that the generalized $(\frac{G'}{G})$-expansion method combined with suitable transformations provides a more powerful tool for constructing exact solutions of the nonlinear Schrödinger equation with variable coefficients and can be used for many other variable-coefficient NLEEs in mathematical physics. Finding suitable transformations and applying the generalized $(\frac{G'}{G})$-expansion method to other variable-coefficient NLEEs in mathematical physics is our task in the future work.

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